On existence of oscillations in Persidskii systems

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Abstract: The conditions of existence of oscillations in the sense of Yakubovich are considered for a class of generalized nonlinear Persidskii systems. To this end, the conditions of local instability at the origin and global boundedness of solutions are presented in the form of linear matrix inequalities. The proposed theory is applied for robustness analysis of nonlinear feedback controls in linear systems with respect to unmodeled dynamics.

1. INTRODUCTION

Stability analysis is a complex issue, therefore, in the theory of control and dynamical systems there are various developments devoted to this problem Khalil (2002). The main approach to check the stability in the nonlinear framework is based on the Lyapunov function method Lyapunov (1992); Malkin (1952); Hahn (1967) and its extensions LaSalle and Lefchetz (1961); van der Schaft (1996); Lin et al. (1996). The shortage of this procedure is the lack of constructive tools assigning a corresponding Lyapunov function to a generic nonlinear system.

In line with stability analysis, studying more complex modes of dynamical systems, including oscillations (periodical or chaotic), attracts the researchers nowadays Fradkov and Pogromsky (1998); Leonov et al. (1995); Martinez et al. (2003); Meglio et al. (2009). A generic and useful theory for studying irregular oscillations has been proposed by V.A. Yakubovich Yakubovich (1973, 1975); Yakubovich and Tomberg (1989). The conditions of oscillations in the sense of Yakubovich are based on existence of two Lyapunov functions Efimov and Fradkov (2009); Efimov and Perruquetti (2015). The first Lyapunov function ensures local instability of an equilibrium, while the second one provides global boundedness of the system trajectories.

In this work a class of Persidskii systems is studied, which have been first introduced in Barbashin (1961) with the Lyapunov functions being a linear combination of the integrals of the nonlinearities, and next by Persidskii in Persidskii (1969), where a combination of the absolute values of the states was used in the Lyapunov function. This class of systems has been widely investigated in the context of diagonal stability Kazkurewicz and Bhaya (1999), neural networks Hopfield and Tank (1986); Sontag (1993), sliding mode controls Aparicio et al. (2019) and in other applications Erickson and Michel (1985). In a recent work Efimov and Aleksandrov (2019), input-to-state stability conditions have been proposed.

The goal of this work is to introduce the conditions of existence of oscillations in the sense of Yakubovich for a class of generalized Persidskii systems (Section 2). For this purpose, the conditions of local instability at the origin and global boundedness of trajectories are obtained in Section 4 (the latter extends the result obtained in Efimov and Aleksandrov (2019)), which can be verified by solving linear matrix inequalities (LMIs). In Section 5 these results are used to investigate robustness of nonlinear control algorithms applied to linear systems, and when an unmodeled dynamics appears in the control channel (such a dynamics, usually, does not allow the system to be stabilized at the origin, then appearance of oscillations implies that at least boundedness of trajectories is preserved).

Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers.
- $|\cdot|$ and $\|\cdot\|$ denote the absolute value in $\mathbb{R}$ and the Euclidean norm on $\mathbb{R}^n$, respectively; for any $\epsilon > 0$ define an open ball around the origin by $\mathcal{B}(\epsilon) = \{x \in \mathbb{R}^n : \|x\| < \epsilon\}$.
- For a (Lebesgue) measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ define the norm $||d||_{[t_0,t_1)} = \text{ess sup}_{t\in[t_0,t_1)}\|d(t)\|$, then $||d||_\infty = ||d||_{[0,+\infty)}$ and the set of d with the property $||d||_\infty < +\infty$ we further denote as $\mathcal{L}_\infty^m$ (the set of essentially bounded measurable functions).

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- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, s) \in \mathcal{K}_\infty$ and $\beta(s, \cdot)$ is a non-increasing function with $\lim_{t\to+\infty} \beta(s,t) = 0$ for each fixed $s \in \mathbb{R}_+$.
- The notation $DV(x)v$ stands for the directional derivative of a continuously differentiable function V evaluated at point x with respect to a vector $v \in \mathbb{R}^n$.
- Denote the identity matrix of dimension $n \times n$ by I_n, the vector of dimension n with all elements equal 1 by $\mathtt{1}_n$, and $\mathrm{diag}\{g\}$ represents a diagonal matrix of dimension $n \times n$ with a vector $g \in \mathbb{R}^n$ on the main diagonal.
- A series of integers $1, 2, ..., n$ is denoted by $\overline{1,n}$.

2. PROBLEM STATEMENT

Consider the following class of extended Persidskii systems:

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^{M} A_j f^j(x(t)) + d(t),\ t \geq 0, \qquad (1)$$

where $x(t) = [x_1(t) \dots x_n(t)]^\top \in \mathbb{R}^n$ is the state vector, $x(0) \in \mathbb{R}^n$; and $d(t) \in \mathbb{R}^n$ is the external perturbation, $d \in \mathcal{L}_\infty^n$; $f^j(x) = [f_1^j(x_1) \dots f_n^j(x_n)]^\top$, $f^j(0) = 0$, $j = \overline{1,M}$ are continuous functions (the solutions of the system (1) exist in the forward time at least locally), the matrices $A_k \in \mathbb{R}^{n\times n}$ for $k = \overline{0,M}$.

Assumption 1. For any $i = \overline{1,n}$, $j = \overline{1,M}$:

$$s f_i^j(s) > 0 \quad \forall s \in \mathbb{R} \setminus \{0\}.$$

In the assumption above, it is stated that all nonlinearities belong to a sector and may take zero values at the origin only. Under this hypothesis, after a proper re-indexing and decomposition of f^j in (1), there exists $m \in \{0, \dots, M\}$ such that for all $i = \overline{1,n}$, $z = \overline{1,m}$:

$$\lim_{s\to\pm\infty} f_i^z(s) = \pm\infty;$$

and there exists $\mu \in \{m, \dots, M\}$ such that for all $i = \overline{1,n}$, $z = \overline{1,\mu}$:

$$\lim_{s\to\pm\infty} \int_0^s f_i^z(\sigma) d\sigma = +\infty.$$

Thus, it is supposed that some of the nonlinearities are radially unbounded, and $m = 0$ corresponds to the case when all nonlinearities are bounded (at least for negative or positive argument). It is also claimed that other part of these nonlinearities may have unbounded integral (clearly if $m > 0$, then all radially unbounded nonlinearities also have unbounded integrals and $\mu \geq m$ due to the introduced sector condition).

If $A_r = 0$ for all $r = \overline{0, M-1}$ and $\mu = M$, then we recover the system studied by Persidskii in the conventional framework Persidskii (1969).

Our goal is to propose conditions establishing a possible presence of oscillating trajectories in (1) (in the sense of Yakubovich, given below).

3. PRELIMINARIES

Consider a nonlinear system:

$$\dot{x}(t) = f(x(t), d(t)),\ t \geq 0, \qquad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^m$ is the external input, $d \in \mathcal{L}_\infty^m$, and $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a locally Lipschitz (or Hölder) continuous function, $f(0,0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$ and input $d \in \mathcal{L}_\infty^m$, define the corresponding solutions by $x(t, x_0, d)$ for any $t \geq 0$ for which the solution exists.

3.1 Basic properties

In this work we will be interested in the following stability properties Khalil (2002); Krasovskii (1963); Hahn (1967); Dashkovskiy et al. (2011):

Definition 1. The system (2) is called *input-to-state practically stable (ISpS)*, if there are some functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and a constant $c \geq 0$ such that

$$\|x(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \gamma(||d||_{[0,t)}) + c \quad \forall t \geq 0$$

for any input $d \in \mathcal{L}_\infty^m$ and any $x_0 \in \mathbb{R}^n$. The function γ is called *nonlinear asymptotic gain.* The system is called *ISS* if $c = 0$.

Definition 2. The system (2) is called *globally practically stable* if there are functions $\sigma, \gamma \in \mathcal{K}$ and a constant $c \geq 0$ such that

$$\|x(t, x_0, d)\| \leq \sigma(\|x_0\|) + \gamma(||d||_{[0,t)}) + c \quad \forall t \geq 0$$

for any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$. For $d = 0$, (2) is called *Lagrange stable,* and it is called just *stable* if $c = 0$.

It is straightforward to conclude that any ISpS (ISS) system (2) is practically stable (stable) with $\sigma(s) = \beta(s, 0)$.

Definition 3. The system (2) is called *unstable* at the origin for $d = 0$, if there is an $\epsilon > 0$ such that for any $\delta > 0$, $x(t', x_0, 0) \notin \mathcal{B}(\epsilon)$ for some $x_0 \in \mathcal{B}(\delta)$ and some $t' \geq 0$.

These properties have the following Lyapunov/Chetaev function characterizations:

Definition 4. A smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called *ISpS Lyapunov function* for the system (2) if for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ and some $r \geq 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\theta \in \mathcal{K}$:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$
$$DV(x)f(x, d) \leq r + \theta(\|d\|) - \alpha_3(\|x\|).$$

Such a function V is called *ISS Lyapunov function* if $r = 0$.

For global practical stability it is enough that an ISpS Lyapunov function exists for $\|x\| > X$ with some $X \in \mathbb{R}_+$.

Definition 5. A smooth function $V : \mathbb{R}^n \to \mathbb{R}$ is called *Chetaev function* for the system (2) with $d = 0$ if $V(0) = 0$, and there exist $\epsilon_0 > 0$ such that $\mathcal{V}^+ \cap \mathcal{B}(\epsilon) \neq \emptyset$ for any $\epsilon \in (0, \epsilon_0]$, where $\mathcal{V}^+ = \{x \in \mathcal{B}(\epsilon_0) : V(x) > 0\}$, and

$$DV(x)f(x, 0) > 0 \quad \forall x \in \mathcal{V}^+.$$

Theorem 6. The system (2) is ISpS (ISS) iff it admits an ISpS (ISS) Lyapunov function.

Theorem 7. The system (2) with $d = 0$ is unstable if it admits a Chetaev function.

Under additional mild restrictions, the existence of a Chetaev function is also necessary for instability Krasovskii (1963); Efimov and Perruquetti (2015).

3.2 Oscillations in the sense of Yakubovich

A function $h : \mathbb{R}^n \to \mathbb{R}$ is called monotone if the condition $x_1 \le x_1', \dots, x_n \le x_n'$ implies that either $h(x_1, \dots, x_n) \le h(x_1', \dots, x_n')$ or $h(x_1, \dots, x_n) \ge h(x_1', \dots, x_n')$.

Definition 8. Yakubovich (1973); Efimov and Fradkov (2009) For $-\infty < \pi^- < \pi^+ < +\infty$ the solution $x(t, x_0, 0)$ of the system (2) with $x_0 \in \mathbb{R}^n$ and $d = 0$ is called $[\pi^-, \pi^+]$–oscillation with respect to an output $h(x)$ (where $h : \mathbb{R}^n \to \mathbb{R}$ is a continuous monotone function) if the solution is defined for all $t \ge 0$ and

$$\liminf_{t\to+\infty} h(x(t, x_0, 0)) = \pi^-, \ \limsup_{t\to+\infty} h(x(t, x_0, 0)) = \pi^+.$$

The solution $x(t, x_0, 0)$ of the system (2) is called oscillating, if there exist some output h and constants π^-, π^+ such that solution $x(t, x_0, 0)$ is $[\pi^-, \pi^+]$–oscillation with respect to the output h. A forward complete system (2) with $d = 0$ is called oscillatory, if for almost all $x_0 \in \mathbb{R}^n$ the solutions $x(t, x_0, 0)$ of the system are oscillating. The oscillatory system (2) is called uniformly oscillatory, if for almost all $x_0 \in \mathbb{R}^n$ for the corresponding solutions $x(t, x_0, 0)$ there exist an output h and constants π^-, π^+ independently on initial conditions.

In other words the solution $x(t, x_0, 0)$ is oscillating if the output $h(x(t, x_0, 0))$ is asymptotically bounded and there is no limit value of $h(x(t, x_0, 0))$ for $t \to +\infty$. Note that the term "almost all solutions" is used to emphasize that generally the system (2) has a nonempty set of equilibria. The notion of oscillations in the sense of Yakubovich is rather generic, it includes periodical oscillations (limit cycles), quasi-periodical, recurrent and chaotic trajectories. An oscillating trajectory in the sense of Yakubovich could be repelling. The trajectories could also be unbounded, it is required to find a function of the state vector, which is bounded and admits certain requirements introduced in Definition 8. Despite its complexity this notion has a simple Lyapunov characterization.

Theorem 9. Efimov and Fradkov (2009) Let the system (2) have two continuously differentiable Lyapunov functions $V_1 : \mathbb{R}^n \to \mathbb{R}_+$ and $V_2 : \mathbb{R}^n \to \mathbb{R}_+$ fulfilling the following inequalities for all $x \in \mathbb{R}^n$:

$$\upsilon_1(\|x\|) \le V_1(x) \le \upsilon_2(\|x\|),$$
$$\upsilon_3(\|x\|) \le V_2(x) \le \upsilon_4(\|x\|), \ \upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4 \in \mathcal{K}_\infty,$$

and for some $0 < X_1 < \upsilon_1^{-1} \circ \upsilon_2 \circ \upsilon_3^{-1} \circ \upsilon_4(X_2) < +\infty$:

$DV_1(x)f(x, 0) > 0$ for all $0 < \|x\| < X_1$ and $x \notin \Pi$;

$DV_2(x)f(x, 0) < 0$ for all $\|x\| > X_2$ and $x \notin \Pi$,

where $\Pi \subset \mathbb{R}^n$ is a set with zero Lebesgue measure containing all equilibria of the system, and

$$\Omega \cap \Pi = \emptyset,$$
$$\Omega = \{x \in \mathbb{R}^n : \upsilon_2^{-1} \circ \upsilon_1(X_1) \le \|x\| \le \upsilon_3^{-1} \circ \upsilon_4(X_2)\}.$$

Then the system (2) is oscillatory.

The Lyapunov function for the linearized system (2) at the origin is a candidate for the function V_1 Yakubovich (1975) (it can be replaced with Chetaev one Efimov and Perruquetti (2015), as in this work). Instead of existence of the function V_2 one can require just boundedness of solutions of the system (2) (if this fact can be verified using another approach not dealing with a Lyapunov function analysis). In Efimov and Fradkov (2009), for a class of uniformly oscillatory systems (2) it is shown that these conditions are also necessary.

3.3 Stability of extended Persidskii systems

In the expressions below, it is assumed that if the upper limit of a summation or a sequence is smaller than the lower one, then the corresponding terms have to be omitted and the conditions avoided.

Theorem 10. Efimov and Aleksandrov (2019) Let Assumption 1 be satisfied and there exist $P = P^\top \in \mathbb{R}^{n\times n}$; $\Xi^k = \mathrm{diag}\{\xi^k\}$ with $\xi^k = [\xi_1^k \dots \xi_n^k]^\top \in \mathbb{R}^n$ for $k = \overline{0, M}$; $\Lambda^j = \mathrm{diag}\{\lambda^j\}$ with $\lambda^j = [\lambda_1^j \dots \lambda_n^j]^\top \in \mathbb{R}^n$ for $j = \overline{1, M}$; $\Upsilon_{s,j} = \mathrm{diag}\{\upsilon^{s,j}\}$ with $\upsilon^{s,j} = [\upsilon_1^{s,j} \dots \upsilon_n^{s,j}]^\top \in \mathbb{R}^n$ for $s = \overline{0, M}$ and $j = \overline{s+1, M}$, and $\Gamma = \Gamma^\top > 0$ such that

$$\Lambda^j \ge 0, \ j = \overline{1, M}; \quad \Xi^k \ge 0, \ k = \overline{0, M};$$
$$P > 0 \text{ or } P \ge 0, \sum_{z=1}^{\mu} \Lambda^z > 0;$$
$$\Upsilon_{s,j} \ge 0, \ s = \overline{0, M}, \ j = \overline{s+1, M}; \tag{3}$$
$$\begin{cases} \Xi^0 > 0 & m = 0 \\ \sum_{k=0}^{m} \Xi^k + 2\sum_{s=0}^{m}\sum_{j=s+1}^{m} \Upsilon_{s,j} > 0 & m > 0 \end{cases}; \ Q \le 0,$$

where

$$Q = \begin{bmatrix} Q_{1,1} & Q_{1,2} & Q_{1,3} & \cdots & Q_{1,M+1} & P \\ Q_{1,2}^\top & Q_{2,2} & Q_{2,3} & \cdots & Q_{2,M+1} & \Lambda^1 \\ Q_{1,3}^\top & Q_{2,3}^\top & Q_{3,3} & \cdots & Q_{3,M+1} & \Lambda^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_{1,M+1}^\top & Q_{2,M+1}^\top & Q_{3,M+1}^\top & \cdots & Q_{M+1,M+1} & \Lambda^M \\ P & \Lambda^1 & \Lambda^2 & \cdots & \Lambda^M & -\Gamma \end{bmatrix}$$
$$Q_{1,1} = A_0^\top P + PA_0 + \Xi^0;$$
$$Q_{j+1,j+1} = A_j^\top \Lambda^j + \Lambda^j A_j + \Xi^j, \ j = \overline{1, M};$$
$$Q_{1,j+1} = PA_j + A_0^\top \Lambda^j + \Upsilon_{0,j}, \ j = \overline{1, M};$$
$$Q_{s+1,j+1} = A_s^\top \Lambda^j + \Lambda^s A_j + \Upsilon_{s,j}, \ s = \overline{1, M}, \ j = \overline{s+1, M}.$$

Then the system (1) is ISS.

The proof of this result is based on the following Lyapunov function:

$$V(x) = x^\top Px + 2\sum_{j=1}^{M}\sum_{i=1}^{n} \lambda_i^j \int_0^{x_i} f_i^j(s)ds, \tag{4}$$

whose time derivative with respect to (1) admits an estimate for $Q \le 0$:

$$DV(x)f(x, d) \le -x^\top \Xi^0 x - \sum_{j=1}^{M} f^j(x)^\top \Xi^j f^j(x)$$
$$-2\sum_{j=1}^{M} x^\top \Upsilon_{0,j} f^j(x) - 2\sum_{s=1}^{M}\sum_{j=s+1}^{M} f^s(x)^\top \Upsilon_{s,j} f^j(x)$$
$$+d^\top \Gamma d. \tag{5}$$

The feature of the LMI (3) consists in a fine treatment of all diagonal items in the off-diagonal blocks of Q represented by the matrices $\Upsilon_{s,j} \ge 0$ for $s = \overline{0, M}$ and $j = \overline{s+1, M}$. Without introduction of $\Upsilon_{s,j}$ all corresponding elements would be considered as perturbations, but actually due to the sector condition imposed on nonlinearities $f^j(x)$ in Assumption 1, these terms can be sign-definite and, hence, helpful.

Remark 11. The condition $\Xi^0 > 0$ of Theorem 10 for the case $m = 0$ can be relaxed to

$$\Xi^0 + 2\sum_{j=1}^{M} \Upsilon_{0,j} > 0$$

under an additional hypothesis that the functions $x_i f_i^j(x_i)$ are radially unbounded for all $i = \overline{1,n}$ and $j = \overline{1,M}$.

4. CONDITIONS OF OSCILLATIONS

In this section, following the ideas given in Theorem 9, first, the conditions of ISpS will be presented extending the result of Theorem 10 by using Theorem 6 and the Lyapunov function (4). Second, the conditions of instability will be given based on Theorem 7. Finally, the conditions of oscillatory behavior will be proposed combining the previously obtained results.

4.1 Global practical stability

To establish global practical stability property we can use the fact that it is related with ISS, whose conditions for (1) are given in Theorem 10. Note that for this property (since it is simpler than ISS) the derivative of the Lyapunov function (4) should be negative definite in x only for sufficiently big values of the state, and even its positive definiteness has to be ensured only outside of a ball. These observations provide an insight how we are going to develop the result of Theorem 10, but before we need to introduce an additional conventions on the shale and ordering of the functions $f^j(x)$, $j = \overline{1,M}$:

Assumption 2. Let

$$\lim_{s\to\pm\infty} \frac{|f_i^j(s)|}{|s|} \in \{0, +\infty\},\ \lim_{s\to\pm\infty} \frac{|f_i^j(s)|}{|f_i^k(s)|} \in \{0, +\infty\}$$

for all $i = \overline{1,n}$ and all $j \neq k = \overline{1,M}$.

The last condition implies that all $f_i^j(x)$, $j = \overline{1,M}$ have nonlinear and different asymptotic growth (this property also can be assured by putting linear components of $f^j(x)$ in the linear part explicitly, the same for the functions of a common rate).

Assumption 3. For any $i = \overline{1,n}$ there exists $Z_i > 0$ such that for all $|s| \geq Z_i$:

$$sf_i^j(s) > 0,\ \forall j = \overline{1,M};$$
$$|f_i^1(s)| \geq |f_i^2(s)| \geq \cdots \geq |f_i^{R_i}(s)| \geq |s|$$
$$\geq |f_i^{R_i+1}(s)| \geq \cdots \geq |f_i^M(s)|,$$

where $R_i \in \{0, \ldots, M\}$ is an index.

In the above inequality, the limit terms f_i^0 and f_i^{M+1} in the cases $R_i = 0$ and $R_i = M$, respectively, have to be omitted. This hypothesis is not restrictive, and in the most cases it can be respected by rearranging and reordering the nonlinearities in (1). For brevity of formulation of the result below define $R = \mathrm{sign}(\min_{1\leq i\leq n} R_i)$, the indices m and μ save their meaning under Assumption 3.

Theorem 12. Let assumptions 2, 3 be satisfied and there exist indices $\kappa \in \{1, \ldots, \mu\}$ and $\iota \in \{R, \ldots, m\}$, the matrices $P = P^\top \in \mathbb{R}^{n\times n}$; $\Xi^k = \mathrm{diag}\{\xi^k\}$ with $\xi^k = [\xi_1^k \ldots \xi_n^k]^\top \in \mathbb{R}^n$ for $k = \overline{0,M}$; $\Lambda^j = \mathrm{diag}\{\lambda^j\}$ with $\lambda^j = [\lambda_1^j \ldots \lambda_n^j]^\top \in \mathbb{R}^n$ for $j = \overline{1,M}$; $\Upsilon_{s,j} = \mathrm{diag}\{\upsilon^{s,j}\}$ with $\upsilon^{s,j} = [\upsilon_1^{s,j} \ldots \upsilon_n^{s,j}]^\top \in \mathbb{R}^n$ for $s = \overline{0,M}$ and $j = \overline{s+1,M}$, and $\Gamma = \Gamma^\top > 0$ such that

$$\Lambda^j \geq 0,\ j = \overline{1,\kappa};$$
$$\begin{cases} P > 0 & \mu = 0 \\ P \geq 0,\ \sum_{z=1}^{\kappa} \Lambda^z > 0 & \mu \geq 1 \end{cases};$$
$$\Xi^s \geq 0,\ \Upsilon_{s,j} \geq 0,\ s = \overline{R,\iota},\ j = \overline{s+1,\iota},$$
$$\begin{cases} \Xi^0 > 0 & m = 0 \\ \sum_{s=R}^{\iota} \Xi^s + 2\sum_{s=R}^{\iota}\sum_{j=s+1}^{\iota} \Upsilon_{s,j} > 0 & m \geq 1 \end{cases};\ Q \leq 0,$$

where the matrix Q is given in Theorem 10. Then the system (1) is globally practically stable. If additionally Assumption 1 holds and

$$\Lambda^j \geq 0,\ j = \overline{1,M},$$

then the system (1) is ISpS.

Proof. The proof is only sketched below due to space limitations, and it is based on the Lyapunov function given in (4). It can be shown that under the conditions of the theorem, the function V outside a ball around the origin is positive and radially unbounded, and its derivative with respect to (1) admits an estimate given in (5), which is negative and radially unbounded with respect to the state x, positive definite with respect to d and has a constant bias (as in Theorem 6 for ISpS).

The results of this subsection introduce a series of tools for analysis of boundedness of trajectories in the extended Persidskii system (1), these tools can be combined and also relaxed considering more particular scenarios and applications.

4.2 Local instability at the origin

For local instability it is necessary to ensure local positive definiteness of V in some directions at the origin and positive definiteness of the derivative $\dot{V}$ also locally in the domain where $V(x) > 0$. To formulate these restrictions let us introduce a reordering of the nonlinearities:

Assumption 4. Let

$$\lim_{s\to 0} \frac{|f_i^j(s)|}{|s|} \in \{0, +\infty\},\ \lim_{s\to 0} \frac{|f_i^j(s)|}{|f_i^k(s)|} \in \{0, +\infty\}$$

for all $i = \overline{1,n}$ and all $j \neq k = \overline{1,M}$.

This condition reads that all $f_i^j(x)$, $j = \overline{1,M}$ have nonlinear and different local growth at the origin.

Assumption 5. There exist $Z > 0$, $\eta > 0$ such that for all $s \in (-Z, 0) \cup (0, Z)$:

$$\eta s f_i^j(s) \geq \int_0^s f_i^j(\sigma)d\sigma > 0,\ \forall j = \overline{1,M};$$
$$|f_i^1(s)| \geq |f_i^2(s)| \geq \cdots \geq |f_i^{R_i}(s)| \geq |s|$$
$$\geq |f_i^{R_i+1}(s)| \geq \cdots \geq |f_i^M(s)|,$$

where $R_i \in \{0, \ldots, M\}$ is an index for all $i = \overline{1,n}$.

As we can see, all restrictions are imposed for $\|x\| < Z$ and there is no requirement on radial unboundedness of functions and integrals for a local investigation.

Theorem 13. Let assumptions 4, 5 be satisfied and there exist indices $\kappa \in \{1, \ldots, M\}$ and $\iota \in \{1, \ldots, n\}$, the matrices $P = P^\top \in \mathbb{R}^{n\times n}$; $\Xi^k = \operatorname{diag}\{\xi^k\}$ with $\xi^k = [\xi_1^k \ldots \xi_n^k]^\top \in \mathbb{R}^n$ for $k = \overline{0, M}$; $\Lambda^j = \operatorname{diag}\{\lambda^j\}$ with $\lambda^j = [\lambda_1^j \ldots \lambda_n^j]^\top \in \mathbb{R}^n$ for $j = \overline{1, M}$; $\Upsilon_{s,j} = \operatorname{diag}\{\upsilon^{s,j}\}$ with $\upsilon^{s,j} = [\upsilon_1^{s,j} \ldots \upsilon_n^{s,j}]^\top \in \mathbb{R}^n$ for $s = \overline{0, M}$ and $j = \overline{s+1, M}$ such that

$$\lambda_\iota^z \geq 0,\ z = \overline{1, \kappa};$$

$$\begin{cases} \sum_{z=1}^{\kappa} \lambda_\iota^z > 0 & \kappa \leq R_\iota \\ P_{\iota,\iota} \geq 0,\ P_{\iota,\iota} + \sum_{z=1}^{\kappa} \lambda_\iota^z > 0 & \kappa > R_\iota \end{cases};$$

$$\Xi^0 + \rho P \leq 0,\ \Xi^j \leq 0,\ \Upsilon_{0,j} + \rho\eta\Lambda^j \leq 0,\ j = \overline{1, M},$$

$$\Upsilon_{s,j} \leq 0,\ s = \overline{1, M},\ j = \overline{s+1, M},$$

$$Q \geq 0,$$

where $\rho > 0$ is a parameter and the matrix Q is given in Theorem 10. Then the system (1) is unstable.

Proof. The proof is again only sketched. By considering the function $V(x)$ given in (4) and calculating a lower estimate for its derivative with respect to (1) for the case $d = 0$, the conditions of Theorem 7 can be verified under the restrictions of the theorem, which implies that the origin is unstable.

4.3 Existence of oscillations

Following the result of Theorem 9, it is straightforward to formulate the required conditions:

Theorem 14. Let the requirements of theorems 12, 13 be satisfied and the system (1) with $d = 0$ admit the only equilibrium at the origin. Then (1) with $d = 0$ is oscillatory.

Proof. Under these restrictions, the set Π from Theorem 9 contains just the origin. Fulfillment of the conditions of Theorem 12 ensures the global boundedness of all trajectories (it gives function V_2 from Theorem 9), while Theorem 13 guarantees instability of the only steady state (it provides a kind of function V_1 from Theorem 9). Hence, there is a compact forward invariant set $\Omega \subset \mathbb{R}^n$ attracting asymptotically almost all trajectories of the system, and obviously $\Omega \cap \Pi = \emptyset$.

Let us demonstrate the efficiency of the developed theory in an application.

5. STABILITY ROBUSTNESS AGAINST UNMODELED DYNAMICS

Consider a linear system:

$$\dot{z}(t) = Az(t) + Bu(t), \tag{6}$$

where $z(t) \in \mathbb{R}^\nu$ and $u(t) \in \mathbb{R}^\mu$ are the state and the control vectors, $\nu > 0$ and $\mu > 0$ are some integers, $A \in \mathbb{R}^{\nu\times\nu}$ and $B \in \mathbb{R}^{\nu\times\mu}$. Assume that there is a control input for this system:

$$u(t) = -\sum_{j=1}^{M} K_j \phi^j(z(t)),$$

where $M > 0$ is an integer and $\phi^j : \mathbb{R}^\nu \to \mathbb{R}^\nu$ are some nonlinear functions, $\phi^j(z) = [\phi_1^j(z_1) \ldots \phi_\nu^j(z_\nu)]^\top$, $\phi^j(0) = 0$, $j = \overline{1, M}$; $K_j \in \mathbb{R}^{\mu\times\nu}$ for $j = \overline{1, M}$; and the closed-loop system is stable and possesses a desired performance (clearly it is a system in the form (1) with $d = 0$). Next, assume that in the control channel there is an unmodeled dynamics (actuator) represented by a linear stable filter:

$$\dot{\zeta}(t) = D\zeta(t) + G\upsilon(t), \tag{7}$$

$$u(t) = C\zeta(t),\ \upsilon(t) = -\sum_{j=1}^{M} K_j \phi^j(z(t)),$$

where $\zeta(t) \in \mathbb{R}^\varpi$ is the state of the filter for an integer $\varpi > 0$, $\upsilon(t) \in \mathbb{R}^\mu$ is the applied to the actuator control designed for the system (6), and $u(t) \in \mathbb{R}^\mu$ is the output, which is the control for the system (6); $C \in \mathbb{R}^{\mu\times\varpi}$, $D \in \mathbb{R}^{\varpi\times\varpi}$ and $G \in \mathbb{R}^{\varpi\times\mu}$. The closed-loop dynamics of (6) and (7) can be rewritten in the form of (1) with $d = 0$:

$$x = [z_1, \ldots, z_\nu, \zeta_1, \ldots, \zeta_\varpi]^\top \in \mathbb{R}^n,\ n = \nu + \varpi,$$

$$f^j(x) = [\phi_1^j(x_1), \ldots, \phi_\nu^j(x_\nu), 0, \ldots, 0]^\top,\ j = \overline{1, M},$$

$$A_0 = \begin{bmatrix} A & BC \\ 0 & D \end{bmatrix},\ A_j = -\begin{bmatrix} 0 \\ G \end{bmatrix} [\, K_j \ 0 \,],\ j = \overline{1, M}.$$

Therefore, using the result of Theorem 14 we can verify the oscillatory property of the closed-loop system under such an unmodeled dynamics of the actuator.

Let $\nu = 1$, $\varpi = 2$ and

$$A = 0,\ B = 1,\ C = [1\ 0],\ D = \begin{bmatrix} -\delta & \delta \\ 0 & -\delta \end{bmatrix},$$

$$G = \begin{bmatrix} 0 \\ \delta \end{bmatrix},\ M = 1,\ \phi^1(z) = |z|^\alpha \operatorname{sign}(z),$$

where $\alpha \in [0, 1)$ and $\delta > 0$ are tuning parameters, $K_1 > 0$. Such a control can be used for finite-time stabilization Bernuau et al. (2014). Hence,

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\delta & \delta \\ 0 & 0 & -\delta \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\delta K_1 & 0 & 0 \end{bmatrix},$$

and the matrix A_0 is not Hurwitz, it has one zero eigenvalue and two equal $-\delta$.

To check the conditions of Theorem 12 select $P = P^\top > 0$, $\lambda^1 = [\lambda_1^1, 0, 0]^\top \in \mathbb{R}^3$ with $\lambda_1^1 > 0$, $\xi^0 = [0, \xi_2^0, \xi_3^0]^\top \in \mathbb{R}^3$ with $\xi_2^0 > 0$, $\xi_3^0 > 0$, and the matrix $\Upsilon_{0,1}$ in a generic form (full matrix) with requirement that

$$(\Upsilon_{0,1})_{1,1} > 0,$$

i.e. the first element of this matrix corresponding to the term $|z|^{\alpha+1}$ is positive. It is obvious that the Lyapunov function $V(x)$ given in (4) is positive definite under introduced restrictions. It is also straightforward to verify that there exist the matrices demanded above such that $Q \leq 0$ and in (5)

$$DV(x)f(x, 0) \leq -(\Upsilon_{0,1})_{1,1} |z|^{\alpha+1} - \xi_2^0 \zeta_1^2$$

$$-\xi_3^0 \zeta_2^2 + 2\left[\left|(\Upsilon_{0,1})_{2,1}\right| |\zeta_1||z|^\alpha + \left|(\Upsilon_{0,1})_{3,1}\right| |\zeta_2||z|^\alpha \right],$$

then the required global boundedness of trajectories is proven. Note that in the linear case, *i.e.* for $\alpha = 1$, some additional restrictions on δ and K_1 have to be imposed to prove the boundedness of solutions.

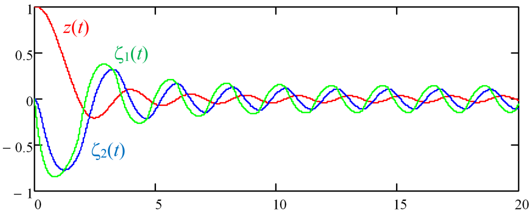

Fig. 1. The results of simulation for $\delta = 3$, $K_1 = 1$ and $\alpha = 0.5$

To use Theorem 13, take any $\vartheta_1 > 0$ and shift the nonlinearity f^1 by linear feedback, then we obtain the new matrix

$$\tilde{A}_0 = A_0 + \vartheta_1 A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\delta & \delta \\ -\vartheta_1 \delta K_1 & 0 & -\delta \end{bmatrix}.$$

It is straightforward to verify that always there exists a value of ϑ_1 sufficiently big such that the matrix $\tilde{A}_0$ is unstable in the direction of the variable z. Since $\alpha \in [0, 1)$, then also for any value of ϑ_1 the function $\tilde{f}_1^j(x) = f_1^j(x_1) - \vartheta_1 x_1$ stays in the required sector onto a vicinity of the origin. Take $P = P^\top > 0$, $\lambda^1 = [\lambda_1^1, 0, 0]^\top \in \mathbb{R}^3$ with $\lambda_1^1 > 0$, $\xi^0 \in \mathbb{R}^3$ with $\xi_1^0 < 0$, and the matrix $\Upsilon_{0,1}$ with requirement that

$$(\Upsilon_{0,1})_{1,1} < 0,$$

then the conditions of the theorem are satisfied with $\iota = 1$.

By Theorem 14 the system oscillatory. The results of simulation for $\delta = 3$, $K_1 = 1$ and $\alpha = 0.5$ are shown in Fig. 1.

6. CONCLUSIONS

For a class of extended Persidskii systems new conditions of ISpS and instability are proposed, which are formulated in terms of LMIs. The proposed theory is combined to predict presence of oscillations in this class of models. These results are used to investigate robustness of nonlinear controls for linear systems with respect to unmodeled dynamics.

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